

New overlap construction of Weyl fermions on the lattice

Christof Gattringer and Markus Pak

Institut für Physik, Universität Graz
8010 Graz, Austria

Abstract

In a recent article Hasenfratz and von Allmen have suggested a fixed point action for two flavors of Weyl fermions on the lattice with gauge group $SU(2)$. The block-spin transformation they use maps the chiral and vector symmetries of the underlying vector theory onto two equations of the Ginsparg-Wilson (GW) type. We show that an overlap Dirac operator can be constructed which solves both GW equations simultaneously. We discuss the properties of this overlap operator and its projection onto lattice Weyl fermions which seems to be free of artefacts, in particular the projection operators are independent of the gauge field.

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Introduction

Lattice gauge theory made a big leap forward when it was understood how the chiral symmetry of a massless vector-like theory manifests itself on the lattice. The lattice Dirac operator has to obey the Ginsparg-Wilson (GW) equation [1] and solutions of this equation are, e.g., the overlap operator [2] and fixed point fermions [3]. A lattice variant of the continuum chiral rotations was constructed and the axial anomaly identified in an elegant way [4].

Immediately after the vector-like case was understood, the problem of a lattice regularization of chiral gauge theories was attacked [5]. The chosen approach was to project a solution of the Ginsparg-Wilson equation for the vector-like theory onto its left-handed components. An unpleasant feature of this projection is that the left-acting projectors and their counterparts acting to the right have a different structure. Due to the GW equation one of the two sets of projectors must depend on the Dirac operator and thus on the gauge field. This leads to unwanted CP violating terms [6, 7] and the problem of having an additional gauge field dependence in the integration measure of the lattice path integral of the chiral gauge theory.

In a recent paper [8] Hasenfratz and von Allmen revisited the problem of transferring continuum symmetries onto the lattice using block spin transformations. In particular they analyzed the connection between anomalies and the blocking prescription used for mapping the continuum theory onto the lattice. The key insight is that the blocking kernel must break all anomalous symmetries of the target theory. For a two-flavor vector-like $SU(2)$ gauge theory in a symmetric representation of the fermion action, they proposed a particular blocking kernel with the correct symmetry breaking pattern. A corresponding fixed point action was derived which obeys two Ginsparg-Wilson type equations for the flavor singlet chiral and vector symmetries of the underlying vector theory. The theory can be projected to left-handed components, and for the resulting theory of Weyl fermions the anomalies were shown to be correctly mapped onto the lattice.

In this paper we use the two Ginsparg-Wilson equations from [8] as a starting point. We show that it is possible to construct a simultaneous overlap solution for both equations. The physical and doubler branches are studied and we establish that the correct continuum limit is approached in the physical sector and the doublers decouple. We discuss different representations of the overlap solution, in particular as the sign function of a hermitian matrix. The overlap operator may be projected to Weyl fermions in a symmetric way without generating artefacts and the measure in the path integral assumes a simple form.

Continuum theory and its symmetries on the lattice

In our construction we start from a vector theory which we later project to a chiral gauge theory. In the continuum the action has the form

$$\begin{aligned} S[\bar{\psi}, \psi] &= \int d^4x \bar{\psi} \gamma_\mu (\vec{\partial}_\mu + iA_\mu) \psi \\ &= \frac{1}{2} \int d^4x \left[\bar{\psi} \gamma_\mu (\vec{\partial}_\mu + iA_\mu) \psi - \psi^T \gamma_\mu^T (\vec{\partial}_\mu + iA_\mu^T) \bar{\psi}^T \right]. \end{aligned} \quad (1)$$

$\bar{\psi}$ and ψ are Grassmann valued Dirac spinors that carry SU(2) color and SU(2) flavor indices which we suppress at the moment but will make them explicit later in the lattice Dirac operators we consider. The gauge field assumes values in su(2), the algebra of our gauge group SU(2).

In the second line of (1) we have already identically rewritten the action in a form that is suitable for identifying the symmetric representation of the fermions which we will use for our lattice discretization. The superscript T denotes transposition. Introducing new 8-component fermion spinors $\Psi^T = (\psi^T, \bar{\psi})$ we can write the fermion action as

$$S[\Psi] = \frac{1}{2} \int d^4x \Psi^T D^{cont} \Psi, \quad D^{cont} = \begin{bmatrix} 0 & -\gamma_\mu^T (\vec{\partial}_\mu + iA_\mu^T) \\ \gamma_\mu (\vec{\partial}_\mu + iA_\mu) & 0 \end{bmatrix}. \quad (2)$$

The continuum Dirac operator D^{cont} has a structure of 4×4 blocks which act on the upper and lower four components of our new spinors Ψ . In flavor space D^{cont} is diagonal.

The Dirac operator contains also the transpose gauge field A_μ^T and for later use we note a relation for this transposition. The su(2)-valued gauge fields can be expressed in terms of the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$,

$$A_\mu = A_\mu^{(1)} \frac{\sigma_1}{2} + A_\mu^{(2)} \frac{\sigma_2}{2} + A_\mu^{(3)} \frac{\sigma_3}{2}, \quad A_\mu^T = A_\mu^{(1)} \frac{\sigma_1}{2} - A_\mu^{(2)} \frac{\sigma_2}{2} + A_\mu^{(3)} \frac{\sigma_3}{2}, \quad (3)$$

and since only σ_2 is anti-symmetric, while σ_1 and σ_3 are symmetric, transposition of the gauge field corresponds to flipping the sign of the second component. Obviously we find

$$A_\mu^T = -\sigma_2 A_\mu \sigma_2 = \varepsilon^c A_\mu \varepsilon^c \quad \text{with} \quad \varepsilon^c = i\sigma_2. \quad (4)$$

The superscript c attached to the ε -tensor refers to the color indices, which ε^c acts on.

Having outlined the details of our continuum target theory, we now need to discuss how symmetries of the continuum theory manifest themselves on the

lattice. In particular we are interested in the flavor singlet chiral and vector transformations of the action (1) which in the notation with the 8-component spinors Ψ are generated by

$$\Gamma_5 = \begin{bmatrix} \mathbb{1}^c \otimes \gamma_5 \otimes \mathbb{1}^f & 0 \\ 0 & \mathbb{1}^c \otimes \gamma_5 \otimes \mathbb{1}^f \end{bmatrix}, \quad \Gamma_V = \begin{bmatrix} \mathbb{1}^c \otimes \mathbb{1}^d \otimes \mathbb{1}^f & 0 \\ 0 & -\mathbb{1}^c \otimes \mathbb{1}^d \otimes \mathbb{1}^f \end{bmatrix}. \quad (5)$$

Here we now have made explicit the action on all involved indices, color, Dirac and flavor, and use the superscripts c, d, f to denote the action in color-, Dirac- and flavor-space. For the upper and lower components of our 8-spinors we will continue to use vector/matrix notation for notational convenience. The corresponding transformations of the 8-component spinors read

$$\Psi \longrightarrow \exp(i\epsilon\Gamma_5) \Psi, \quad \Psi \longrightarrow \exp(i\epsilon\Gamma_V) \Psi. \quad (6)$$

The action (2) is invariant since D^{cont} anti-commutes both with Γ_5 and Γ_V .

The lattice manifestation of continuum symmetries is most conveniently obtained by analyzing the symmetries of a block spin transformation,

$$e^{-\frac{1}{2}\Phi^T D \Phi} = \int \mathcal{D}[\Psi] e^{-(\Phi - \Psi^B)^T E^{-1} (\Phi - \Psi^B)} e^{-S[\Psi]}. \quad (7)$$

Here Ψ^B denotes a blocked field living on the lattice, which is obtained from its continuum counterpart through a suitable gauge covariant blocking prescription. The fields of the lattice theory are denoted by Φ and the corresponding lattice Dirac operator is D . The blocked continuum fields Ψ^B and the lattice fields Φ are coupled through a blocking kernel E^{-1} in the first exponent on the rhs. The blocking kernel is diagonal in the discrete lattice space-time indices and we use vector/matrix notation for the lattice indices, both in the term that couples the blocked and lattice fields, as well as for the quadratic form $\Phi^T D \Phi$ in the lattice action on the lhs. On the rhs. the continuum field Ψ is integrated over in a path integral to eliminate the degrees of freedom above the lattice cutoff.

Exploring the invariance of the continuum action it is possible to map the continuum symmetries onto the lattice. They manifest themselves (see e.g. [1, 9, 10] for a derivation) in Ginsparg-Wilson equations for the lattice Dirac operator D , and since here we are interested in both, the vector and the chiral symmetry, we obtain two such equations (a denotes the lattice spacing)

$$\begin{aligned} \Gamma_5 D + D \Gamma_5 &= a D E \Gamma_5 D, \\ \Gamma_V D + D \Gamma_V &= a D E \Gamma_V D. \end{aligned} \quad (8)$$

The continuum transformations (6) change to the corresponding symmetry transformations of the lattice fields (which we denote only up to $\mathcal{O}(\epsilon)$),

$$\Phi \longrightarrow \Phi + i\epsilon \Gamma (\mathbb{1} - aED/2) \Phi + \mathcal{O}(\epsilon^2) \quad , \quad \Gamma = \Gamma_5, \Gamma_V \quad , \quad (9)$$

which change the integration measure of the lattice fields according to

$$\mathcal{D}[\Psi] \longrightarrow \mathcal{D}[\psi] \left(1 + i\epsilon \text{Tr} [\Gamma (\mathbb{1} - aED/2)] + \mathcal{O}(\epsilon^2) \right) \quad , \quad (10)$$

a relation that may be used to identify the anomalies.

An important issue is the choice of the blocking kernel E^{-1} , which couples the blocked and the lattice fields in the block-spin transformation (7). The central construction principle [8] is that one must break all the symmetries which are anomalous in the quantized target theory, while other, non-anomalous symmetries need not, but may be broken if it is convenient to do so. Thus for the lattice version of a vector-like theory it is sufficient to break the U(1) chiral symmetry, as was done in [1] with a blocking matrix proportional to the identity. Since here we are ultimately interested in the chiral version of (1), e.g., the case where we project to left-handed fermions, we need to break both the U(1) chiral and the U(1) vector symmetries. The choice suggested in [8] reads

$$E = i \begin{bmatrix} \varepsilon^c \otimes \overline{C} \otimes \varepsilon^f & 0 \\ 0 & \varepsilon^c \otimes \overline{C} \otimes \varepsilon^f \end{bmatrix} \quad . \quad (11)$$

The matrix E commutes with both generators Γ_5 and Γ_V and thus has the above discussed symmetry breaking pattern (a vanishing anti-commutator would correspond to an unbroken symmetry).

In E tensors ε^c and ε^f for color and flavor are used. Already introduced above, we here list some properties of these ε -tensors which we will need later,

$$\varepsilon = i\sigma_2 \quad , \quad \varepsilon = -\varepsilon^T = -\varepsilon^\dagger = -\varepsilon^{-1} \quad . \quad (12)$$

In Dirac space E applies a modified charge conjugation matrix \overline{C} which is related to the conventional charge conjugation matrix C . We here use the chiral representation where C and \overline{C} obey

$$\begin{aligned} C &= i\gamma_2\gamma_4 \quad , \quad \overline{C} = i\gamma_5 C \quad , \quad \overline{C}C = C\overline{C} = i\gamma_5 \quad , \\ C &= -C^T = C^\dagger = C^{-1} \quad , \quad \overline{C} = -\overline{C}^T = -\overline{C}^\dagger = -\overline{C}^{-1} \quad . \end{aligned} \quad (13)$$

Both, C and \overline{C} relate the γ_μ matrices to their transpose,

$$\gamma_\mu^T = -C\gamma_\mu C \quad , \quad \gamma_\mu^T = -\overline{C}\gamma_\mu \overline{C} \quad . \quad (14)$$

From (12) and (13) follow the properties of E ,

$$E = -E^T = E^\dagger = E^{-1} \quad , \quad E\Gamma_5 = \Gamma_5 E \quad , \quad E\Gamma_V = \Gamma_V E \quad . \quad (15)$$

Overlap solution for the GW equations

We now show that a simultaneous overlap solution of the two Ginsparg-Wilson equations (8) can be constructed.

We begin with examining the transposition properties of the $SU(2)$ valued link variables in the fundamental representation, which are used for introducing the gauge field on the lattice. An element U of $SU(2)$ may be written as

$$U = \exp(i\vec{A} \cdot \vec{\sigma}/2) = \mathbb{1}^c \cos((\vec{A}/2)^2) + i \frac{\vec{A} \cdot \vec{\sigma}/2}{(\vec{A}/2)^2} \sin((\vec{A}/2)^2) \quad , \quad (16)$$

with a real valued coefficient vector \vec{A} . Similar to the algebra valued continuum field in (4), we may write the transpose of the gauge link with the help of the ε -tensor $\varepsilon^c = i\sigma_2$,

$$U^T = \sigma_2 U^\dagger \sigma_2 = -\varepsilon^c U^\dagger \varepsilon^c \quad . \quad (17)$$

The construction of the overlap solution makes use of a generalized Wilson Dirac operator which is built from the naive discretization $V_\mu(x, y)$ of the covariant derivative and the Wilson term $S(x, y)$ which is proportional to the covariant Laplace operator,

$$\begin{aligned} V_\mu(x, y) &= \frac{1}{2} [U_\mu(x) \delta_{x+\hat{\mu}, y} - U_\mu(x - \hat{\mu})^\dagger \delta_{x-\hat{\mu}, y}] \quad , \\ S(x, y) &= 4\mathbb{1}^c \delta_{x, y} - \frac{1}{2} \sum_{\mu=1}^4 [U_\mu(x) \delta_{x+\hat{\mu}, y} + U_\mu(x - \hat{\mu})^\dagger \delta_{x-\hat{\mu}, y}] \quad . \end{aligned} \quad (18)$$

The two terms, are anti-hermitian and hermitian, respectively,

$$V_\mu^\dagger = -V_\mu \quad , \quad S^\dagger = S \quad . \quad (19)$$

Using the relation (17) their transposition properties may be expressed as

$$V_\mu^T = \varepsilon^c V_\mu \varepsilon^c \quad , \quad S^T = -\varepsilon^c S \varepsilon^c \quad . \quad (20)$$

The generalized Wilson Dirac operator D_W which we use for the overlap is constructed from V_μ and S ,

$$D_W = \frac{1}{a} \begin{bmatrix} i\varepsilon^c S \otimes \overline{C} \otimes \varepsilon^f & -V_\mu^T \otimes \gamma_\mu^T \otimes \mathbb{1}^f \\ V_\mu \otimes \gamma_\mu \otimes \mathbb{1}^f & iS\varepsilon^c \otimes \overline{C} \otimes \varepsilon^f \end{bmatrix} \quad . \quad (21)$$

The derivatives V_μ are arranged such that in the naive continuum limit they approach the continuum Dirac operator D^{cont} as given in (2). The term S , which removes the doublers, couples to E , as can be seen by comparing the blocks on the diagonal in (21) to (11).

The overlap solution D we present here is given by

$$\begin{aligned} D &= \frac{1}{a} \left[E - A (E \Gamma_5 A E \Gamma_5 A)^{-1/2} \right] \\ &= \frac{1}{a} \left[E - A (E \Gamma_V A E \Gamma_V A)^{-1/2} \right], \end{aligned} \quad (22)$$

where

$$A = E - a D_W. \quad (23)$$

In (22) we have displayed the overlap operator in two different forms. The fact that the two forms are identical follows from the relation

$$E \Gamma_5 A E \Gamma_5 = E \Gamma_V A E \Gamma_V, \quad (24)$$

which may be established using the identities (12) – (14). Using for each of the two GW equations (8) the suitable form of D , the equations (8) reduce to

$$\begin{aligned} E \Gamma_5 A (E \Gamma_5 A E \Gamma_5 A)^{-1/2} E \Gamma_5 A (E \Gamma_5 A E \Gamma_5 A)^{-1/2} &= \mathbb{1}, \\ E \Gamma_V A (E \Gamma_V A E \Gamma_V A)^{-1/2} E \Gamma_V A (E \Gamma_V A E \Gamma_V A)^{-1/2} &= \mathbb{1}. \end{aligned} \quad (25)$$

These two identities hold trivially as can be seen using the spectral theorem for the inverse square root. This establishes that (22) solves both Ginsparg-Wilson equations (8).

The Dirac operator may be brought into a second form, using the sign function of a hermitian matrix. This is possible due to the fact that the products $E \Gamma_5 A$ and $E \Gamma_V A$ both are hermitian matrices,

$$(E \Gamma_5 A)^\dagger = E \Gamma_5 A, \quad (E \Gamma_V A)^\dagger = E \Gamma_V A. \quad (26)$$

These equations may be established using (12) – (14) and (20). From (26) it follows that the arguments of the inverse square roots in (22) are squares of hermitian matrices. Using $(E \Gamma_5)^2 = \mathbb{1}$ and $(E \Gamma_V)^2 = \mathbb{1}$, we find the sign representation of the overlap operator

$$D = \frac{1}{a} \left[E - E \Gamma_5 \text{sign}(E \Gamma_5 A) \right] = \frac{1}{a} \left[E - E \Gamma_V \text{sign}(E \Gamma_V A) \right]. \quad (27)$$

Having established, that the overlap operator (22) solves the two Ginsparg-Wilson equations (8), we still need to show that it gives rise to the correct

continuum limit and removes the doublers. We begin this analysis with the physical branch where V_μ and S behave as

$$\begin{aligned} V_\mu &= a \left[\vec{\partial}_\mu + iA_\mu \right] + \mathcal{O}(a^2) , \\ V_\mu^T &= a \left[\overleftarrow{\partial}_\mu + iA_\mu^T \right] + \mathcal{O}(a^2) , \\ S &\propto \mathcal{O}(a^2) . \end{aligned} \quad (28)$$

Inserting these into (21) and subsequently in (22), one finds with the help of (12) – (14) and (20) that in the physical branch our overlap operator approaches the correct continuum operator D^{cont} as given in (2),

$$D = D^{cont} + \mathcal{O}(a) . \quad (29)$$

For the doubler branches one has

$$V_\mu \propto \mathcal{O}(a) , \quad S = \mathbb{1}^c 2k + \mathcal{O}(a^2) , \quad k = 1, 2, 3, 4 , \quad (30)$$

and inserting these again into (21) and (22) gives rise to the behavior

$$D = \frac{2}{a} E + \mathcal{O}(1) . \quad (31)$$

The rhs. diverges as $a \rightarrow 0$ and thus the doublers decouple. It is interesting to note, that the term S , which removes the doublers, couples to the blocking matrix E , which has eigenvalues $+1$ and -1 . Consequently the doubler modes end up symmetrically at the positions $\pm 2/a$ in the complex plane. This is different from the usual vector-like overlap operator, where the doublers all end up on the positive real axis near $2/a$. We stress, however, that this is not a peculiarity of the overlap solution given here. Also the fixed point solution of [8], which for the free case can be computed in closed form following [11], distributes the doublers symmetrically.

Finally it is interesting to observe, that when going back to the conventional notation with 4-spinors, the term that removes the doublers assumes the form

$$i \frac{2}{a} \left[\psi^T \epsilon^c \otimes \overline{C} \otimes \epsilon^f \psi + \overline{\psi} \epsilon^c \otimes \overline{C} \otimes \epsilon^f \overline{\psi}^T \right] . \quad (32)$$

The same structure is obtained for the free fixed point operator computed by direct blocking from the continuum. We stress that such a term cannot be formulated within the usual bilinear representation of the fermion action, and is possible only in the symmetric form used here.

Properties of D and its projection to Weyl fermions

In this section we now discuss some of the key properties of D which are necessary for the discussion [8] of the anomalies of our theory and its projection to Weyl fermions.

The analysis of the anomaly given in [8] makes use of the two GW equations (8) and the fact that the Dirac operator D is $\hat{\Gamma}_5$ -hermitian, i.e., it obeys

$$\hat{\Gamma}_5 D \hat{\Gamma}_5 = D^\dagger, \quad (33)$$

where

$$\hat{\Gamma}_5 = \begin{bmatrix} 0 & \mathbb{1}^c \otimes \gamma_5 \otimes \mathbb{1}^f \\ \mathbb{1}^c \otimes \gamma_5 \otimes \mathbb{1}^f & 0 \end{bmatrix}. \quad (34)$$

For our overlap operator Eq. (33) can be shown by noting that

$$\hat{\Gamma}_5 A \hat{\Gamma}_5 = A^\dagger = E \Gamma_5 A E \Gamma_5, \quad (35)$$

where the last identity is a consequence of (15) and (26). Thus the overlap Dirac operator can also be written as

$$D = \frac{1}{a} \left[E - A \left(\hat{\Gamma}_5 A \hat{\Gamma}_5 A \right)^{-1/2} \right] = \frac{1}{a} \left[E - \hat{\Gamma}_5 \text{sign} \left(\hat{\Gamma}_5 A \right) \right]. \quad (36)$$

The $\hat{\Gamma}_5$ -hermiticity of D then follows from this equation together with the $\hat{\Gamma}_5$ -hermiticity of A and $\hat{\Gamma}_5^2 = \mathbb{1}$.

For a Dirac operator that obeys (8) and (33) it was shown in [8] that the correct anomaly structure of a vector-like theory emerges, i.e., the vector transformation is free of anomalies, while for the flavor singlet chiral transformation the correct axial anomaly is found.

Having fully understood the vector-like theory on the lattice, we can now turn to the chiral gauge theory. For our symmetric representation of the fermions, we define suitable left- (P_-) and right-handed (P_+) projectors,

$$P_\pm = \begin{bmatrix} \mathbb{1}^c \otimes \frac{\mathbb{1}^d \pm \gamma_5}{2} \otimes \mathbb{1}^f & 0 \\ 0 & \mathbb{1}^c \otimes \frac{\mathbb{1}^d \mp \gamma_5}{2} \otimes \mathbb{1}^f \end{bmatrix}. \quad (37)$$

The projectors obey

$$P_\pm P_\pm = P_\pm, \quad P_\pm P_\mp = 0, \quad P_+ + P_- = \mathbb{1}, \quad P_\pm^T = P_\pm, \quad (38)$$

which are the usual properties of projectors and their symmetry under transposition which we stress explicitly as this is important for the symmetric fermion representation used here.

The Dirac operator D_- for left-handed Weyl fermions is then obtained by projecting the vector-like operator D from, e.g, the right,

$$D_- = D P_- = P_- D . \quad (39)$$

In this equation we have already noted that the projection may also be done from the left (with the same projector). The fact that D and the projectors commute is an important consistency relation which follows from the possibility to write the projectors as

$$P_{\pm} = \frac{1}{2} \left[\mathbb{1} \pm \Gamma_V \Gamma_5 \right] , \quad (40)$$

together with the two GW equations (8). We finally remark that also the Weyl operator D_- obeys the GW equations, as is expected from the corresponding symmetry in the continuum [10].

For the projected operator D_- it was shown in [8] that the correct fermion number anomaly is obtained. The arguments are again only based on the GW equations (8) and the $\hat{\Gamma}_5$ -hermiticity (33). Since our overlap operator obeys all of these, the proof of [8] applies and we conclude that the projected overlap operator has the correct chiral anomaly. At the same time the projection operators are independent of D and thus no unwanted gauge field dependence is introduced in the path integral measure of the chiral theory.

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References

- [1] P. H. Ginsparg and K. G. Wilson, Phys. Rev. D 25, 2649 (1982).
- [2] R. Narayanan and H. Neuberger, Nucl. Phys. B 443 (1995) 305; H. Neuberger, Phys. Lett. B 417 (1998) 141.
- [3] P. Hasenfratz and F. Niedermayer, Nucl. Phys. B 414 (1994) 785 [arXiv:hep-lat/9308004]; P. Hasenfratz, S. Hauswirth, K. Holland, T. Jörg, F. Niedermayer and U. Wenger, Int. J. Mod. Phys. C 12 (2001) 691 [arXiv:hep-lat/0003013].
- [4] M. Lüscher, Phys. Lett. B 428 (1998) 342 [arXiv:hep-lat/9802011].

- [5] M. Lüscher, Nucl. Phys. B 549 (1999) 295 [arXiv:hep-lat/9811032], Nucl. Phys. B 568 (2000) 162 [arXiv:hep-lat/9904009], JHEP 0006 (2000) 028 [arXiv:hep-lat/0006014], arXiv:hep-th/0102028; H. Neuberger, Phys. Rev. D 63 (2001) 014503 [arXiv:hep-lat/0002032]; H. Suzuki, Prog. Theor. Phys. 101 (1999) 1147 [arXiv:hep-lat/9901012], Nucl. Phys. B 585 (2000) 471 [arXiv:hep-lat/0002009]; H. Igarashi, K. Okuyama and H. Suzuki, arXiv:hep-lat/0012018; H. Suzuki, JHEP 0010 (2000) 039 [arXiv:hep-lat/0009036]; D. Kadoh and Y. Kikukawa, arXiv:0709.3656 [hep-lat], arXiv:0709.3658 [hep-lat].
- [6] P. Hasenfratz, Nucl. Phys. Proc. Suppl. 106, 159 (2002) [arXiv:hep-lat/0111023].
- [7] P. Hasenfratz and M. Bissegger, Phys. Lett. B 613 (2005) 57 [arXiv:hep-lat/0501010].
- [8] P. Hasenfratz and R. von Allmen, JHEP 0802 (2008) 079 [arXiv:0710.5346 [hep-lat]].
- [9] P. Hasenfratz, F. Niedermayer and R. von Allmen, JHEP 0610 (2006) 010 [arXiv:hep-lat/0606021].
- [10] C. Gattringer and M. Pak, PoS LAT2007, 081 (2007) [arXiv:0710.5371 [hep-lat]].
- [11] W. Bietenholz and U. J. Wiese, Nucl. Phys. B 464, 319 (1996) [arXiv:hep-lat/9510026].